Navier-Stokes-alpha model: LES equations with nonlinear dispersion

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Abstract

We present a framework for discussing LES equations with nonlinear dispersion. In this framework, we discuss the properties of the nonlinearly dispersive Navier-Stokes-alpha (NS$-\alpha$) model of incompressible fluid turbulence — also called the viscous Camassa-Holm equations in the literature — in comparison with the corresponding properties of large eddy simulation (LES) equations obtained via the approximate-inverse approach. In this comparison, we identify the spatially filtered NS$-\alpha$ equations with a class of generalized LES similarity models. Applying a certain approximate inverse to this class of LES models restores the Kelvin circulation theorem for the defiltered velocity and shows that the NS$-\alpha$ model describes the dynamics of the defiltered velocity for this class of generalized LES similarity models. We also show that the subgrid scale forces in the NS$-\alpha$ model transform covariantly under Galilean transformations and under a change to a uniformly rotating reference frame. Finally, we discuss in the spectral formulation how the NS$-\alpha$ model retains the local interactions among the large scales ; retains the nonlocal sweeping effects of large scales on small scales ; yet attenuates the local interactions of the small scales amongst themselves.

1 Introduction

The effects of subgrid-scale (SGS) fluid motions occurring below the available resolution of numerical simulations must be modeled. One way of modeling these effects is simply to discard the energy that reaches such subgrid scales. This is clearly unacceptable, though, and many creative alternatives have been offered. A prominent example is the large eddy simulation (LES) approach, see, e.g., [1] - [4]. The LES approach is based on applying a

spatial filter to the Navier-Stokes equations. The reduction of flow complexity and information content achieved in the LES approach depends on the characteristics of the filter that one uses, its type and width. In particular, the LES approach introduces a length scale into the description of fluid dynamics, namely, the width of the filter used. The LES approach is conceptually different from the Reynolds Averaged Navier-Stokes (or, RANS) approach, which is based on statistical arguments and exact ensemble averages, rather than spatial and temporal filtering. After filtering, however, just as in the RANS approach, one faces the classic turbulence closure problem: How to model the effects of the filtered-out subgrid scales in terms of the remaining resolved fields? In practice, this problem is compounded by the requirement that the equations be solved numerically, thereby introducing further approximations.

A turbulence modeling scheme — called here the Navier-Stokes-alpha model, or NS$-\alpha$ model (also called the viscous Camassa-Holm equations in [5] - [8]) — was recently introduced. This model imposes an energy "penalty" inhibiting the creation of smaller and smaller excitations below a certain length scale (denoted alpha), but still allows for nonlinear sweeping of the smaller scales by the larger ones. This energy penalty implies a ***nonlinearly dispersive modification*** of the Navier-Stokes equations. The alpha-modification appears in the nonlinear convection term, it depends on length scale and we emphasize that it is dispersive, not dissipative. This modification causes the translational kinetic energy wavenumber spectrum of the NS$-\alpha$ model to roll off rapidly below the length scale alpha as k^{-3} in three dimensions, instead of continuing to follow the slower Kolmogorov scaling law, $k^{-5/3}$, thereby shortening its inertial range and making it more computable [10].

This roll-off in the energy spectrum means that the inertial range is curtailed and the length scale ℓ_α at which viscous dissipation takes over in the NS$-\alpha$ model is *larger* than for the Navier-Stokes equations with the same viscosity. Hence, for a given driving force and viscosity, the number of active degrees of freedom N^α_{dof} for the NS$-\alpha$ model is *smaller* than for Navier-Stokes. A rigorous analytical estimate was derived in [8] for the fractal dimension D_{frac} of the *global attractor* for the NS$-\alpha$ model. Namely,

$$D_{frac} \le (N^\alpha_{dof})^{3/2} \quad \text{and} \quad N^\alpha_{dof} \equiv (L/\ell_\alpha)^3 \simeq \frac{L}{\alpha} Re^{3/2}\,, \tag{1}$$

where L is the integral scale (or domain size), ℓ_α is the end of the NS$-\alpha$ inertial range and $Re = L^{4/3}\varepsilon_\alpha^{1/3}/\nu$ is the Reynolds number (with NS$-\alpha$ energy dissipation rate ε_α and viscosity ν). The Kolmogorov dissipation length scale is $\ell_{Ko} = (\nu^3/\varepsilon_\alpha)^{1/4}$. The three length scales $\alpha > \ell_\alpha > \ell_{Ko}$ are related by

$$\ell_\alpha^3 \simeq \alpha\, \ell_{Ko}^2 \quad \text{or} \quad \frac{\alpha}{\ell_\alpha} \simeq \frac{\ell_\alpha^2}{\ell_{Ko}^2}\,. \tag{2}$$

Thus, these length scales $\alpha > \ell_\alpha > \ell_{Ko}$ stand in relation to each other in the same way as the length scales $(K^3/\varepsilon^2)^{1/2} > (K\nu/\varepsilon)^{1/2} > (\nu^3/\varepsilon)^{1/4}$ in a K-epsilon model of Navier-Stokes turbulence.

The number of degrees of freedom for a corresponding Navier-Stokes flow with the same viscosity and energy dissipation rate is

$$N_{dof}^{NS} \equiv (L/\ell_{Ko})^3 \simeq Re^{9/4}\,, \tag{3}$$

The implication of these estimates of degrees of freedom for direct numerical simulations that access a significant number of them using the NS$-\alpha$ model is an increase in computational speed relative to Navier-Stokes of

$$\left(\frac{N_{dof}^{NS}}{N_{dof}^{\alpha}}\right)^{4/3} \simeq \left(\frac{\alpha}{L}\right)^{4/3} Re\,. \tag{4}$$

Thus, if α tends to a constant value, say $L/100$, when the Reynolds number increases – as found in [5] - [7] by comparing steady NS$-\alpha$ solutions with experimental data for turbulent flows in pipes and channels – then one could expect to obtain a substantial increase in computability by using the NS$-\alpha$ model at high Reynolds numbers. An early indication of the reliability of using these estimates to gain a relative increase in computational speed in direct numerical simulations of homogeneous turbulence in a periodic domain is given in [9].

In this paper, we introduce the NS$-\alpha$ model as one of a class of equations obtained by filtering the Kelvin circulation theorem for Navier-Stokes. These equations possess the vortex stretching and transport properties emphasized in [10]. We then compare a spatially filtered version of the NS$-\alpha$ equations with a generalization of the LES similarity models. We also calculate the kinematic tranformation properties of the NS$-\alpha$ model under changes of reference frames and discuss its nonlinear spectral-mode dynamics.

2 Kelvin-filtered turbulence models

Following [10], the Navier-Stokes-alpha (NS$-\alpha$) model may be introduced by starting from Kelvin's circulation theorem for the original Navier-Stokes (NS) equations,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = \nu\nabla^2\mathbf{v} + \mathbf{f}\,, \quad \text{with} \quad \nabla\cdot\mathbf{v} = 0\,, \tag{5}$$

for an appropriate forcing $\mathbf{f}$ and constant kinematic viscosity ν. These equations satisfy ***Kelvin's circulation theorem***,

$$\frac{d}{dt}\oint_{\gamma(\mathbf{v})} \mathbf{v}\cdot d\mathbf{x} = \oint_{\gamma(\mathbf{v})} (\nu\nabla^2\mathbf{v} + \mathbf{f})\cdot d\mathbf{x}\,, \tag{6}$$

for a fluid loop $\gamma(\mathbf{v})$ that moves with velocity $\mathbf{v}(\mathbf{x},t)$, the Eulerian fluid velocity.

Kelvin-filtering the Navier-Stokes equations. The equations for the NS$-\alpha$ model may be introduced by modifying the NS Kelvin circulation theorem (6) to integrate around a

loop $\gamma(\hat{\mathbf{v}})$ that moves with a ***spatially filtered Eulerian fluid velocity*** given by $\hat{\mathbf{v}} = g*\mathbf{v}$, where $*$ denotes the convolution,

$$\hat{\mathbf{v}} = g * \mathbf{v} = \int g(\mathbf{x} - \mathbf{x}')\, \mathbf{v}(\mathbf{x}')\; d^3x' \equiv L_g \mathbf{v}(\mathbf{x})\,. \tag{7}$$

The "inverse" operation is denoted

$$\mathbf{v} = \mathcal{O}\hat{\mathbf{v}} \equiv L_g^{-1}\hat{\mathbf{v}}\,, \tag{8}$$

thereby defining an operator $\mathcal{O}$ whose Green's function is the filter g and which we shall assume is positive, symmetric, isotropic, translation-invariant and time-independent. Under these assumptions the quantity (kinetic energy)

$$E = \frac{1}{2}\int \mathbf{v}\cdot\hat{\mathbf{v}}\; d^3x = \frac{1}{2}\int \mathbf{v}\cdot g * \mathbf{v}\; d^3x = \frac{1}{2}\int \hat{\mathbf{v}}\cdot\mathcal{O}\hat{\mathbf{v}}\; d^3x\,, \tag{9}$$

defines a norm.

We modify the Navier-Stokes equations (5) by replacing in their Kelvin's circulation theorem (6) the loop $\gamma(\mathbf{v})$ with another loop $\gamma(\hat{\mathbf{v}})$ moving with the spatially filtered velocity, $\hat{\mathbf{v}}$. Then we have,

$$\frac{d}{dt}\oint_{\gamma(\hat{\mathbf{v}})} \mathbf{v}\cdot d\mathbf{x} = \oint_{\gamma(\hat{\mathbf{v}})} (\nu\nabla^2\mathbf{v} + \mathbf{f})\cdot d\mathbf{x}\,. \tag{10}$$

After taking the time derivative inside the Kelvin loop integral moving with filtered velocity $\hat{\mathbf{v}}$ and reconstructing the gradient of pressure, we find the ***Kelvin-filtered Navier-Stokes equation***,

$$\frac{\partial\mathbf{v}}{\partial t} + \hat{\mathbf{v}}\cdot\nabla\mathbf{v} + \nabla\hat{\mathbf{v}}^{\mathrm{T}}\cdot\mathbf{v} + \nabla p = \nu\nabla^2\mathbf{v} + \mathbf{f}\,, \tag{11}$$

with

$$\nabla\cdot\hat{\mathbf{v}} = 0\,, \quad \text{and} \quad \mathbf{v} = \mathcal{O}\hat{\mathbf{v}}\,. \tag{12}$$

The transport velocity $\hat{\mathbf{v}}(\mathbf{x}, t)$ is the spatially filtered Eulerian fluid velocity in Eq. (7). Note that the continuity equation is now imposed as $\nabla\cdot\hat{\mathbf{v}} = \nabla\cdot(g*\mathbf{v}) = 0$. If filtering commutes with differentiation, then $g*(\nabla\cdot\mathbf{v})$ also vanishes and the transported velocity $\mathbf{v}$ is also divergenceless. The energy balance relation derived from the NS$-\alpha$ equation (11) is

$$\frac{d}{dt}\int \frac{1}{2}\hat{\mathbf{v}}\cdot\mathbf{v}\; d^3x = \int \hat{\mathbf{v}}\cdot\mathbf{f}\; d^3x - \nu\int \mathrm{tr}(\nabla\hat{\mathbf{v}}^T\cdot\nabla\mathbf{v})\; d^3x\,, \tag{13}$$

where we have used incompressibility and have dropped any boundary terms that appear upon integrating by parts.

Vortex transport and stretching. Let $\mathbf{q} = \nabla\times\mathbf{v}$ be the vorticity of the unfiltered Eulerian fluid velocity. The curl of the Kelvin-filtered Navier-Stokes equation (11) gives the vortex transport and stretching equation,

$$\frac{\partial\mathbf{q}}{\partial t} + \hat{\mathbf{v}}\cdot\nabla\mathbf{q} - \mathbf{q}\cdot\nabla\hat{\mathbf{v}} = \nu\nabla^2\mathbf{q} + \nabla\times\mathbf{f}\,. \tag{14}$$

We note that the coefficient $\nabla\hat{\mathbf{v}}$ in the vortex stretching term is the gradient of the ***spatially filtered*** Eulerian velocity $\hat{\mathbf{v}}$. Thus, Kelvin-filtering tempers the vortex stretching in the modified Navier-Stokes equation (11), while preserving the original form of the vortex dynamics. This tempered vorticity stretching is reminicent of Leray's approach in regularizing the Navier-Stokes equations. Except for the term $(\nabla\hat{\mathbf{v}})^{\mathrm{T}}\cdot\mathbf{v}$, the Kelvin-filtered Navier-Stokes equation (11) is otherwise quite similar to Leray's regularization of the Navier-Stokes equations proposed in 1934 [11]. Extension of the Leray regularization to satisfy the Kelvin circulation theorem was cited as an outstanding problem in Gallavotti's review [12]. Indeed, for the case that $\mathcal{O}$ is the Helmholtz operator, full advantage of this regularized vorticity stretching effect was taken in [8] to prove the global existence and uniqueness of strong solutions for the three dimensional NS$-\alpha$ model. Paper [8] also proves the estimate (1) for the fractal dimension of the global attractor for the NS$-\alpha$ model.

The ***Navier-Stokes-alpha model*** emerges from the Kelvin-filtered Navier-Stokes equation (11) when one chooses the operator $\mathcal{O}$ to be the Helmholtz operator, thereby introducing a constant α that has dimensions of length,

$$\mathcal{O} = 1-\alpha^2\nabla^2\,, \quad \text{with} \quad \alpha = const\,. \tag{15}$$

In this case, the filtered and unfiltered fluid velocities in Eq. (8) are related by

$$\mathbf{v} = (1-\alpha^2\nabla^2)\hat{\mathbf{v}}\,, \tag{16}$$

and the reconstructed pressure p is related to the hydrodynamic pressure P as

$$p = P - \frac{1}{2}|\hat{\mathbf{v}}|^2 - \frac{\alpha^2}{2}|\nabla\hat{\mathbf{v}}|^2\,. \tag{17}$$

The original derivation of the ideal ***Euler-alpha model*** (the $\nu=0$ case of the NS$-\alpha$ model) was obtained by using the Euler-Poincaré approach in [13], [14]. The physical interpretations of $\hat{\mathbf{v}}$ and $\mathbf{v}$ as the Eulerian and Lagrangian mean fluid velocities were given in [15].

The kinetic energy norm corresponding to (9) for the NS$-\alpha$ model is given by

$$E_\alpha = \int\Big[\frac{1}{2}|\hat{\mathbf{v}}|^2 + \frac{\alpha^2}{2}|\nabla\hat{\mathbf{v}}|^2\Big]d^3x\,. \tag{18}$$

This kinetic energy is the sum of a translational kinetic energy based on the spatially filtered Eulerian velocity $\hat{\mathbf{v}}$, and a gradient-velocity kinetic energy, multiplied by α^2. By showing the global boundedness, in time, of the kinetic energy (18) one concludes that the coefficient $\nabla\hat{\mathbf{v}}$ in the vortex stretching relation (14) is *bounded in* L^2 for the NS$-\alpha$ model. Thus, the second term in the kinetic energy (18) imposes an energy penalty for creating small scales. The spatial integral of $|\nabla\hat{\mathbf{v}}|^2$ in the second term has the same dimensions as filtered enstrophy (the spatial integral of $|\nabla\times\hat{\mathbf{v}}|^2$, the squared filtered vorticity). For a domain with boundary, or when considering transformations to rotating frames, the spatial integral of $\frac{1}{2}|\nabla\hat{\mathbf{v}}|^2$ in the second term in the kinetic energy norm (18) should be replaced by the integral of trace $(\hat{\mathbf{e}}\cdot\hat{\mathbf{e}})$,

where $\hat{\mathbf{e}}$ is the strain rate tensor, $\hat{\mathbf{e}} = (1/2)(\nabla\hat{\mathbf{v}} + \nabla\hat{\mathbf{v}}^{\mathrm{T}})$ of the filtered velocity in Euclidean coordinates [16]. For the case we treat here, in Euclidean coordinates and in the absence of boundaries, all three of these norms are equivalent for an incompressible flow.

3 Comparison of the NS$-\alpha$ model with LES equations

The large eddy simulation (LES) equations for an incompressible flow are conventionally written as

$$\frac{\partial}{\partial t}\,\overline{u}_i + \frac{\partial}{\partial x_j}\,\overline{u}_i\,\overline{u}_j = -\frac{\partial}{\partial x_i}\,\overline{P} + \nu\frac{\partial^2}{\partial x_j \partial x_j}\,\overline{u}_i - \frac{\partial}{\partial x_j}\,\tau_{i\,j} \tag{19}$$

$$\text{with} \qquad \frac{\partial\,\overline{u}_i}{\partial\,x_i} = 0. \tag{20}$$

The overbar denotes spatial filtering which, for a quantity $f(\mathbf{x})$, is defined by the integral

$$\overline{f}(\mathbf{x}) = \int G(\mathbf{x}-\mathbf{x}'\,)\,f(\mathbf{x}'\,)\,d^3x' \equiv L_G f(\mathbf{x})\,, \tag{21}$$

where G is a given kernel function, not necessarily the same as g in Eq. (7). The quantities in Eq. (19), u_i, P, and ν are the velocity components, pressure, and the kinematic viscosity, respectively, and the effects of subgrid-scale quantities on the resolved velocity are described by a subgrid-scale (SGS) stress tensor

$$\tau_{i\,j} = \overline{u_i u_j} - \overline{u}_i\,\overline{u}_j, \tag{22}$$

which must be modeled in terms of the resolved quantities to close Eq. (19). Note that the unfiltered velocity u_i is assumed to be a solution of the incompressible Navier-Stokes equation.

In contrast, the NS$-\alpha$ equation (11) with definition (16) and (17) is written in index notation as

$$\frac{\partial}{\partial t}v_i + \hat{v}_j\frac{\partial v_i}{\partial x_j} + v_k\frac{\partial \hat{v}_k}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(P - \frac{1}{2}\hat{v}_k\hat{v}_k - \frac{\alpha^2}{2}\frac{\partial \hat{v}_m}{\partial x_l}\frac{\partial \hat{v}_m}{\partial x_l}\right) + \nu\frac{\partial^2}{\partial x_j\partial x_j}v_i, \tag{23}$$

where $\hat{v}_i$ is the filtered quantity related to the NS$-\alpha$ velocity v_i through the equation

$$v_i = \hat{v}_i - \alpha^2\frac{\partial^2}{\partial x_j \partial x_j}\hat{v}_i = \hat{v}_i - \alpha^2\nabla^2\hat{v}_i \tag{24}$$

and α is the filter length scale. Both fields v_i and $\hat{v}_i$ are incompressible since the 'hat' filter commutes with differentiation for a constant alpha.

The NS$-\alpha$ equation (23) can be rearranged into a form similar to the LES equation (19)

$$\frac{\partial}{\partial t}v_i + \frac{\partial}{\partial x_j}v_i v_j = -\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2}{\partial x_j \partial x_j}v_i - \frac{\partial}{\partial x_j}\left((\hat{v}_j - v_j)v_i - \alpha^2\frac{\partial \hat{v}_k}{\partial x_j}\frac{\partial \hat{v}_k}{\partial x_i}\right). \tag{25}$$

Two interpretations of the NS$-\alpha$ velocity. Both the NS$-\alpha$ velocity v_i and the *filtered* NS$-\alpha$ velocity $\hat{v}_i$ can be represented numerically using a lower resolution than required to represent the full NS velocity for the same problem. This is the property that they share with the filtered velocity in traditional LES. Nevertheless, it must be recognized from the derivation in the previous section that there is no LES filtering procedure that would produce the NS$-\alpha$ equations from the full NS equations. Nevertheless because of the similar behavior both NS$-\alpha$ velocities can be formally interpreted as the traditional LES velocity. If the NS$-\alpha$ velocity v_i in equation (25) is interpreted as the filtered Navier-Stokes velocity $\overline{u}_i$ in the LES equation (19) then the subgrid-stress tensor associated with the alpha model is

$$\tau^{\alpha}_{ij} = (\hat{v}_j - v_j)v_i - \alpha^2 \frac{\partial \hat{v}_k}{\partial x_j}\frac{\partial \hat{v}_k}{\partial x_i} = -\,\alpha^2\left(\frac{\partial \hat{v}_k}{\partial x_j}\frac{\partial \hat{v}_k}{\partial x_i} - \hat{v}_i \nabla^2 \hat{v}_j\right) - \alpha^4 (\nabla^2 \hat{v}_i)(\nabla^2 \hat{v}_j) \tag{26}$$

where the last equality is obtained using Eq. (24). This is a straightforward identification of terms.

Alternatively, one may propose to interpret the *filtered* NS$-\alpha$ velocity $\hat{v}_i$ as the filtered Navier-Stokes velocity $\overline{u}_i$. To obtain the corresponding SGS stress in this case we first filter the NS$-\alpha$ equation (25) with the 'hat' filter. Since this filtering commutes with differentiation one finds

$$\frac{\partial}{\partial t}\hat{v}_i + \frac{\partial}{\partial x_j}\hat{v}_i\hat{v}_j = -\frac{\partial \hat{P}}{\partial x_i} + \nu \frac{\partial^2}{\partial x_j \partial x_j}\hat{v}_i - \frac{\partial}{\partial x_j}\left(\left(\widehat{\hat{v}_i v_j} - \hat{v}_i\hat{v}_j\right) - \alpha^2\left(\widehat{\frac{\partial \hat{v}_k}{\partial x_j}\frac{\partial \hat{v}_k}{\partial x_i}} + \hat{v}_j\widehat{\nabla^2}\hat{v}_i\right)\right), \tag{27}$$

which involves doubly filtered quantities. The corresponding SGS stress tensor is

$$\begin{aligned}
\tau^{f\alpha}_{ij} &= (\widehat{\hat{v}_i v_j} - \hat{v}_i\hat{v}_j) - \alpha^2\left(\widehat{\frac{\partial \hat{v}_k}{\partial x_j}\frac{\partial \hat{v}_k}{\partial x_i}} + \hat{v}_j\widehat{\nabla^2}\hat{v}_i\right) && (28)\\
&= (\widehat{\hat{v}_i \hat{v}_j} - \hat{\hat{v}}_i\hat{\hat{v}}_j) - \alpha^2 \widehat{\frac{\partial \hat{v}_k}{\partial x_j}\frac{\partial \hat{v}_k}{\partial x_i}} - \alpha^2\left(\hat{v}_j\widehat{\nabla^2}\hat{v}_i - \hat{\hat{v}}_j\nabla^2\hat{\hat{v}}_i\right) && (29)\\
&\quad + \alpha^2\hat{\hat{v}}_i\nabla^2\hat{\hat{v}}_j - \alpha^4(\nabla^2\hat{\hat{v}}_i)(\nabla^2\hat{\hat{v}}_j),
\end{aligned}$$

where the last equality is obtained using Eq. (24). We shall next consider the transformation properties of the NS$-\alpha$ model. In section 5 we shall discuss how the SGS stress tensor $\tau^{f\alpha}_{ij}$ for the filtered NS$-\alpha$ model is related to LES similarity models for NS.

4 Transformation properties

Galilean invariance. Because of the first term in Eq. (26), or equivalently the expression $\hat{v}_i\nabla^2\hat{v}_j$ in the second part of (26), the SGS tensor τ^{α}_{ij} is neither symmetric nor Galilean invariant, in contrast to the ordinary SGS stress tensor (22). However, the SGS force which

is the divergence of the SGS stress tensor is Galilean invariant for both representations of τ^{α}_{ij}. Therefore, the dynamics of the alpha model is Galilean invariant. Similarly, the SGS stress tensor $\tau^{f\alpha}_{ij}$ is not symmetric and lacks the Galilean invariance because of the term $\hat{\hat{v}}_i\nabla^2\hat{\hat{v}}_j$ but the Galilean invariance is restored in its divergence since $\hat{\hat{v}}_j$ is incompressible.

Rotating frame. It is of interest to consider a transformation of the SGS stress tensor to a frame of reference rotating with a uniform angular velocity Ω_i. The full SGS stress tensor (22) transforms in such a way that the SGS force, i.e. $\partial\tau_{ij}/\partial x_j$, in an inertial and in a rotating frame are the same. Horiuti [17] has recently analyzed several SGS models and showed that some of them do not satify this condition.

The velocity in a rotating frame, denoted by the asterisk, and the velocity in an inertial frame are related as follows

$$u_i = u^*_i + \epsilon_{imn}\Omega_m x^*_n \, . \tag{30}$$

Transforming the substantial derivative to a rotating frame and using (30) gives

$$\frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x_j}u_i u_j = \frac{\partial}{\partial t}u^*_i + \frac{\partial}{\partial x^*_j}u^*_i u^*_j + 2\epsilon_{imn}\Omega_m u^*_n + \epsilon_{imn}\epsilon_{nkl}\Omega_m\Omega_k x^*_l, \tag{31}$$

where the last two terms are the ***Coriolis force and the centrifugal force***, respectively. Transforming the NS$-\alpha$ equation (25) to a rotating frame gives

$$\frac{\partial}{\partial t}v^*_i + \frac{\partial}{\partial x^*_j}v^*_i v^*_j = -2\epsilon_{imn}\Omega_m v^*_n - \epsilon_{imn}\epsilon_{nkl}\Omega_m\Omega_k x^*_l - \frac{\partial P}{\partial x^*_i} + \nu\frac{\partial^2}{\partial x^*_j\partial x^*_j}v^*_i - \frac{\partial}{\partial x^*_j}(\tau^{\alpha *}_{ij} + \tau^{c*}_{ij}), \tag{32}$$

where $\tau^{\alpha *}_{ij}$ is the expression (26) for 'star' quantities and the additional correction term appears

$$\tau^{c*}_{ij} = \alpha^2\left(\epsilon_{imn}\Omega_m x^*_n\nabla^2\hat{v}^*_j - \epsilon_{kmj}\Omega_m\frac{\partial\hat{v}^*_k}{\partial x^*_i} - \epsilon_{kli}\Omega_l\frac{\partial\hat{v}^*_k}{\partial x^*_j}\right). \tag{33}$$

The divergence of this correction stress tensor does not vanish; in fact,

$$\frac{\partial}{\partial x^*_j}\tau^{c*}_{ij} = \alpha^2\left(2\epsilon_{imn}\Omega_m\nabla^2\hat{v}^*_n - \epsilon_{kmj}\Omega_m\frac{\partial^2\hat{v}^*_k}{\partial x^*_i\partial x^*_j}\right). \tag{34}$$

However, the first term in (34) combines with the Coriolis force in Eq. (32) and the second term combines with the centrifugal force to yield the following form of the NS$-\alpha$ equation in the rotating frame:

$$\frac{\partial}{\partial t}v^*_i + \frac{\partial}{\partial x^*_j}v^*_i v^*_j = -2\epsilon_{imn}\Omega_m\hat{v}^*_n - \frac{\partial\Pi}{\partial x^*_i} + \nu\frac{\partial^2}{\partial x^*_j\partial x^*_j}v^*_i + \frac{\partial}{\partial x^*_j}\tau^{\alpha *}_{ij}, \tag{35}$$

where the total pressure Π is

$$\Pi = P + \frac{1}{2}(\Omega_m x_m^*)(\Omega_l x_l^*) - \frac{1}{2}(\Omega_m \Omega_m)(x_l^* x_l^*) - \alpha^2 \epsilon_{mjk} \Omega_m \frac{\partial \hat{v}_k^*}{\partial x_j^*} \,. \tag{36}$$

Remarks. This slight pressure shift is of no importance for incompressible fluid dynamics. Had we replaced $\frac{1}{2}|\nabla\hat{\mathbf{v}}|^2$ in the second term of the kinetic energy norm (18) by trace $(\hat{\mathbf{e}} \cdot \hat{\mathbf{e}})$, as mentioned earlier, we would have removed this pressure shift at the cost of making the motion equation and stress tensor slightly more complicated. We shall decline this option here; however, it shall be *required* in the compressible case.

Note that in the transformed motion equation (35) the Coriolis force is computed using *filtered* velocity $\hat{\mathbf{v}}^*$. This is the physical velocity of the fluid parcels in the rotating frame.

One may also obtain the *nondissipative part* of the NS$-\alpha$ motion equation written in a rotating frame as in (35) by first substituting $\hat{v}_i = \hat{v}_i^* + \epsilon_{imn}\Omega_m x_n^*$ into Hamilton's principle in which the Lagrangian is given by the kinetic energy (18) and then taking variations in the Euler-Poincaré framework, as discussed in [14].

5 Relation to generalized similarity models

We shall discuss the relation of the NS$-\alpha$ model to other SGS models in terms of the inversion of the filtering operation (21). In generalized similarity models for the NS equations the full NS velocity u_i in the expression for the SGS stress tensor (22) is obtained by an inversion of the filtering operation (21), sometimes called ***deconvolution***. The deconvolution is frequently encountered in image processing and its properties are discussed in many textbooks on this subject, e.g. [18, 19]. In general, the inversion of the continuous filter is a ill-conditioned and often a singular problem precluding a unique solution. This difficulty leads to a concept of an approximate or regularized deconvolution such as a pseudo-inverse through a singular value decomposition approach [18]. In the context of subgrid-scale modeling the concept of deconvolution was discussed previously by several authors. Germano [20] introduced an exponential filter for which the filtering operation can be inverted exactly, i.e. the function $f(x)$ in (21) can be represented as a closed form expression in terms of the filtered function $\overline{f}(x)$. Using this representation one obtains a formal expression for the subgrid-scale stress tensor as a function of the filtered field. Germano [20] shows that for this filter the dominant term in the expression for the subgrid scale stress has the form of a nonlinear velocity gradient model

$$\tau_{ij} \sim \frac{\partial \overline{u}_i}{\partial x_k}\frac{\partial \overline{u}_j}{\partial x_k}. \tag{37}$$

This form, considered first by Leonard [21], has been more recently investigated by Liu *et al.* [22], Borue and Orszag [23], Leonard [24], and Winckelmans *et al.* [25]. In addition to the exact deconvolution procedure of Germano [20] and Leonard [24] other approaches

have been proposed and utilized recently. Geurts [26] constructs an ***approximate inverse operator*** by requiring that polynomials up to a certain order are recovered exactly from their filtered counterparts. Stolz and Adams [27] use a formal power series expansion of the inverse operator and obtain the approximate inverse by truncating the expansion. Similar methods are used by Horiuti [17] leading to a multi-level filtered model. The deconvolution is also an integral part of the SGS estimation model of Domaradzki *et al.* [28, 29, 30]. In the spectral space version of the estimation model an exact deconvolution is possible [28, 30]. In the physical space representation the approximate deconvolution is accomplished by solving a sequence of tridiagonal equations [29].

Here, following Stolz and Adams [27] and Horiuti [17] we shall consider a formal inverse of the linear operator L_G in (21) as a power series expansion

$$L_G^{-1} = (I - (I - L_G))^{-1} = I + (I - L_G) + (I - L_G)^2 + ... \tag{38}$$

where I is the identity operator. To first order in the expansion (38) the velocity u_i is approximated as

$$u_i = L_G^{-1} * \overline{u}_i \approx \overline{u}_i + (\overline{u}_i - \overline{\overline{u}}_i)\,. \tag{39}$$

The second term in the approximation (39) is obtained by filtering of the difference $(u_i - \overline{u}_i)$ which can be written using the Taylor expansion as

$$\begin{aligned} u_i - \overline{u}_i &= \int (u_i(\mathbf{x}) - u_i(\mathbf{x}+\mathbf{x}'))\, G(\mathbf{x}')\, d^3x' && (40)\\ &= \int \left(-\frac{\partial u_i}{\partial x_j} x'_j - \frac{1}{2}\frac{\partial^2 u_i}{\partial x_k \partial x_m} x'_k x'_m + \ldots \right) G(\mathbf{x}')\, d^3x' \approx -\frac{\Delta^2}{24}\nabla^2 u_i(\mathbf{x}) && (41) \end{aligned}$$

where the derivatives are evaluated at $\mathbf{x}$ and the above relation is valid for both the symmetric top hat filter or the Gaussian filter with the width Δ in each Cartesian direction. Using this relation the approximate inversion (39) becomes

$$u_i \approx \overline{u}_i - \frac{\Delta^2}{24}\nabla^2 \overline{u}_i\,, \tag{42}$$

upon again invoking commutation of filtering and differentiation. By comparing (42) for the approximate inversion and (24) for the definition of $\hat{v}_i$ in the NS$-\alpha$ model, we recognize that v_i and $\hat{v}_i$ are related by the approximate inverse (39) with $\alpha^2 = \Delta^2/24$. We shall pursue this relation farther by comparing the NS$-\alpha$ model with LES generalized similarity models.

Employing the approximate inverse (42) in the expression for the SGS stress tensor (22) one finds

$$\begin{aligned} \tau_{ij}^{sim} &= \overline{u_i u_j} - \overline{u}_i\, \overline{u}_j \approx (\overline{\bar{u}_i \bar{u}_j} - \bar{\bar{u}}_i \bar{\bar{u}}_j) \\ &\quad - \alpha^2 \left[(\overline{\bar{u}_j \nabla^2 \bar{u}_i} - \bar{\bar{u}}_j \nabla^2 \bar{\bar{u}}_i) + (\overline{\bar{u}_i \nabla^2 \bar{u}_j} - \bar{\bar{u}}_i \nabla^2 \bar{\bar{u}}_j) \right] \\ &\quad + \alpha^4 (\overline{\nabla^2 \bar{u}_i \nabla^2 \bar{u}_j} - \nabla^2 \bar{\bar{u}}_i \nabla^2 \bar{\bar{u}}_j)\,. \end{aligned} \tag{43}$$

The generalized similarity tensor τ_{ij}^{sim} in expression (43) is symmetric and Galilean invariant. On comparing the last equation with the expression for $\tau_{ij}^{f\alpha}$ in Eq. (29), we find as the most prominent difference the product derivative term in the filtered alpha-model equation

$$\tau_{ij}^{f\alpha} = \tau_{ij}^{\alpha sim} - \alpha^2 \left(\widehat{\frac{\partial \hat{v}_k}{\partial x_j} \frac{\partial \hat{v}_k}{\partial x_i}} - \hat{v}_i \widehat{\nabla^2} \hat{v}_j \right) - \alpha^4 (\nabla^2 \widehat{\hat{v}_i)(}\nabla^2 \hat{v}_j) = \tau_{ij}^{\alpha sim} + \widehat{\tau_{ij}^{\alpha}}, \tag{44}$$

where $\tau_{ij}^{\alpha sim}$ is expression (43) written in terms of v_i and the 'hat' filter, instead of u_i and the 'bar' filter. Also $\widehat{\tau}_{ij}^{\alpha}$ is the 'hat'-filtered version of τ_{ij}^{α} in Eq. (26). According to Eq. (44) the SGS stress tensor for the filtered NS$-\alpha$ equation (27) is the sum of the similarity stress tensor for NS and the filtered SGS stress tensor (26) for the original NS$-\alpha$ equation (25). The last equality is the ***Germano identity*** [31] and follows from the the definitions of $\tau_{ij}^{\alpha sim}$, τ_{ij}^{α}, and $\tau_{ij}^{f\alpha}$.

We note that the important new term appearing in all SGS stress tensor expressions for the alpha-model, namely the product of derivatives, is different from the similar product of derivatives in the nonlinear models (37).

The NS-α model is reminiscent, but not equivalent to the generalized similarity models. This is encouraging because it may avoid difficulties encountered by the pure similarity models. One of the most serious problems encountered by the pure similarity models is insufficient SGS dissipation caused by neglecting in the modeling scales which cannot be resolved on a given LES grid. This fact has been long recognized, e.g. by Zhou *et al.* [32], and more recently by Domaradzki and Loh [29]. Indeed, consider the most favorable situation when the filtering operation can be inverted exactly, i.e. the function $f(x)$ in (21) can be represented as a closed form expression in terms of the filtered function $\overline{f}(x)$. In such a case one immediately obtains an expression for the subgrid-scale stress tensor as a function of the filtered field from the definition (22). However, the apparent simplicity of such an exact approach to subgrid-scale modeling is deceptive. This is because in the procedure $f(x)$ and $\overline{f}(x)$ have the same spectral support since the same LES mesh is used to represent both functions. Therefore, no modeling of the subgrid scales is actually performed in the deconvolution procedure. As a result the similarity SGS stress tensor is unable to account for the dissipative effects of the nonlinear interactions between the resolved scales and the unknown, subgrid scales which cannot be represented on a given LES mesh. This implies that a good SGS model should either explicitly model subgrid scales or possess extra terms in addition to the similarity part. The former approach is used in the SGS estimation modeling [28, 29, 30] and the additional terms are encountered in the mixed models [33, 34, 35] as well as in the NS-α model (44).

Section 5 summary. We have identified the 'hat'-filtered NS$-\alpha$ equations with a class of generalized LES similarity models in which the SGS stress tensor possesses additional terms involving products of derivatives, as in $\widehat{\tau}_{ij}^{\alpha}$. Applying the approximate-inverse methods of deconvolution to this class of models restores the Kelvin circulation theorem to the deconvolved solution. The corresponding dynamics of the deconvolved solution is the NS$-\alpha$ model, whose analytical and numerical properties are summarized in [10].

6 Spectral space interpretation

The nonlinear term in the NS$-\alpha$ equation (25) transforms to spectral (Fourier) space as

$$F.T.\left[\frac{\partial}{\partial x_j}v_l(\mathbf{x})\hat{v}_j(\mathbf{x})\right] = ik_j\int v_l(\mathbf{p})\hat{v}_j(\mathbf{k}-\mathbf{p})d^3p = ik_j\int v_l(\mathbf{k}-\mathbf{p})\hat{v}_j(\mathbf{p})d^3p\,. \tag{45}$$

Here the spectral representation is indicated by the dependence on wavenumbers. The incompressibility condition requires projection of (45) onto a plane normal to the vector $\mathbf{k}$ which is accomplished by applying the tensor

$$P_{lm}(\mathbf{k}) = (\delta_{lm} - k_l k_m/k^2). \tag{46}$$

In spectral space one has

$$v_j(\mathbf{k}) = \hat{v}_j(\mathbf{k}) + \alpha^2 k^2 \hat{v}_j(\mathbf{k}), \tag{47}$$

so the projection of the nonlinear term can be written as

$$\frac{1}{2}i\left(k_j P_{lm}\int d^3p\,\frac{v_j(\mathbf{p})}{1+\alpha^2p^2}v_m(\mathbf{k}-\mathbf{p}) + k_m P_{lj}\int d^3p\,v_j(\mathbf{p})\frac{v_m(\mathbf{k}-\mathbf{p})}{1+\alpha^2|\mathbf{k}-\mathbf{p}|^2}\right). \tag{48}$$

This expression allows us to interpret the alpha-model in terms of triad interactions. Qualitatively, the factor $1/(1+\alpha^2k^2)$ attenuates amplitudes of modes with wavenumbers $k > 1/\alpha$ and leaves modes with $k < 1/\alpha$ largely unaffected.

Consider first a mode with $k < 1/\alpha$. Its triadic interactions with all other modes $\mathbf{p}$ and $\mathbf{q} = \mathbf{k} - \mathbf{p}$ for which $p, q < 1/\alpha$ are only slightly affected by the attenuation factors and so are essentially the same as for the Navier-Stokes dynamics. If one of the wavenumbers p, q is greater than $1/\alpha$ its effect on the mode $\mathbf{k}$ is weakened by one of the attenuation factors in (48). In particular, the highly nonlocal interactions of the eddy viscosity type, $k < 1/\alpha$ and $p, q \gg 1/\alpha$, are removed from the dynamics.

Consider now a mode with $k > 1/\alpha$. In this case all local triads with $p, q \sim O(k)$ are attenuated and the nonlocal triads with $p, q \gg 1/\alpha$ are again removed from the dynamics. However, not all nonlocal triads affecting the mode $\mathbf{k}$ are neglected. Indeed, for $k > 1/\alpha$ it is possible to have triads with $p < 1/\alpha < k$ and $q \sim O(k)$ and then at least one of the terms in (48) is not directly affected by the attenuation factors. The NS$-\alpha$ model thus retains *nonlocal sweeping effects* of large scales (either p, or $q < 1/\alpha < k$) acting on small scales $k > 1/\alpha$.

Section 6 summary. The NS$-\alpha$ model is equivalent to the Navier-Stokes dynamics for all triads with $k, p, q < 1/\alpha$, it consistently neglects the nonlocal interactions of the eddy viscosity type for modes $p, q > 1/\alpha$, and retains nonlocal sweeping effects of large eddies on small eddies (p or $q < 1/\alpha < k$).

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